

# Self-accelerating cosmologies and hairy black holes in ghost-free bigravity and massive gravity

Mikhail S. Volkov

Laboratoire de Mathématiques et Physique Théorique CNRS-UMR 7350,  
Université de Tours, Parc de Grandmont, 37200 Tours, FRANCE;

Institut des Hautes Etudes Scientifiques (IHES), 91440 Bures-sur-Yvette, FRANCE  
E-mail: [volkov@lmpt.univ-tours.fr](mailto:volkov@lmpt.univ-tours.fr)

**Abstract.** We present a survey of the known cosmological and black hole solutions in ghost-free bigravity and massive gravity theories. These can be divided into three classes. First, there are solutions with proportional metrics, which are the same as in General Relativity with a cosmological term, which can be positive, negative or zero. Secondly, for spherically symmetric systems, there are solutions with non-bidiagonal metrics. The  $g$ -metric fulfills Einstein equations with a positive cosmological term and a matter source, while the  $f$ -metric is anti-de Sitter. The third class contains solutions with bidiagonal metrics, and these can be quite complex. The time-dependent solutions describe homogeneous (isotropic or anisotropic) cosmologies which show a late-time self-acceleration or other types of behavior. The static solutions describe black holes with a massive graviton hair, and also globally regular lumps of energy. None of these are asymptotically flat. Including a matter source gives rise to asymptotically flat solutions which exhibit the Vainshtein mechanism of recovery of General Relativity in a finite region.

## 1. Introduction

The recent discovery of a multiparameter class of ghost-free massive gravity theories by de Rham, Gabadadze and Tolley (dRGT) [1], and of its bigravity generalization by Hassan and Rosen (HR) [2], has revived the old idea that gravitons can have a small mass  $m$  [3] (see [4] for a recent review). Theories with massive gravitons were, for a long time, considered as pathological, mainly because they exhibit the Boulware-Deser (BD) ghost – an unphysical negative norm state in the spectrum [5]. However, there are serious reasons to believe [6, 7, 8, 9, 10, 11] that the theories of [1, 2] are free of this pathology. Hence, they can be considered as healthy models, and it is interesting to explore their physical consequences.

Theories with massive gravitons have been used in order to explain the observed acceleration of our universe [12, 13]. This acceleration could be accounted for by introducing a cosmological term in Einstein equations, however, this would pose the problem of explaining the origin and value of this term. An alternative possibility is to consider modifications of General Relativity (GR), and theories with massive gravitons are natural candidates for this, since the graviton mass can effectively manifest itself as a small cosmological term [14]. This motivates studying cosmological and other solutions with massive gravitons.

Theories with massive gravitons are described by two metrics,  $g_{\mu\nu}$  and  $f_{\mu\nu}$ . In massive gravity theories the  $f$ -metric is non-dynamical and is usually chosen to be flat, although other choices are also possible, while the dynamical  $g$ -metric describes massive gravitons. In bigravity theories both metrics are dynamical and describe together two gravitons, one of which is massive and the other one is massless. The theory contains two gravitational couplings,  $\kappa_g$  and  $\kappa_f$ , and in the limit where  $\kappa_f$  vanishes, the  $f$ -equations decouple from  $g$ , so that the  $f$ -metric is determined by its own dynamics. The  $g$ -equations contain  $f$ , and the  $g$ -theory then can be viewed as a massive gravity with a fixed background metric  $f$ . Therefore, massive gravity theory is a special case of bigravity, and so when one studies bigravity solutions one finds, in particular, all massive gravity ones. One should emphasize, however, that in the  $\kappa_f \rightarrow 0$  limit the  $f$ -metric does not necessarily become flat and the bigravity solutions do not always reduce in this limit to solutions of the standard dRGT massive gravity with flat  $f$ .

The known bigravity solutions can be divided into three types. First, there are solutions for which the two metrics are proportional in the same coordinate system:  $f_{\mu\nu} = C^2 g_{\mu\nu}$ . This is only possible if  $C$  fulfills an algebraic equation with constant coefficients, and if the matter sources in the two sectors are fine-tuned to be proportional to each other. The  $g$ -equations then reduce to the standard Einstein equations with a matter source and a cosmological term  $\Lambda_g \sim m^2$ , which can be positive, negative or zero. Therefore, one obtains in this way all GR solutions, in particular, for  $C = 1$ , those of vacuum GR. For  $\Lambda_g > 0$  there are cosmological solutions which approach the de Sitter space at late times and describe the cosmic self-acceleration. On the other hand, none of the proportional solutions, apart from the trivial one with  $f_{\mu\nu} = g_{\mu\nu} = \eta_{\mu\nu}$ , fulfill

equations of the massive gravity theory with flat  $f$ .

Secondly, imposing spherical symmetry, there are solutions described by two metrics which are not bidiagonal, although they can be made separately diagonal when expressed in two different frames. There is a nontrivial consistency condition relating these two frames, and as soon as it is fulfilled, the two metrics formally decouple one from the other and each of them fulfills its own set of Einstein equations with its own cosmological and matter terms. In some cases, the  $f$ -metric can be chosen to be anti-de Sitter, with  $\Lambda_f \sim \kappa_f^2$ , hence it becomes flat when  $\kappa_f \rightarrow 0$ , and so the dRGT massive gravity is naturally recovered.

These solutions exist both in the dRGT massive gravity and in the HR bigravity theories. They describe all known massive gravity black holes, in which case the  $g$ -metric is Schwarzschild-de Sitter. They also describe all known massive gravity cosmologies, for which the  $g$ -metric is of any (open, closed and flat) standard Friedmann-Lemâitre-Robertson-Walker (FLRW) type. For these solutions the ordinary matter dominates at early times, when the universe is small, while later the effective cosmological term  $\Lambda_g \sim m^2$  becomes dominant, leading to a self-acceleration. Such solutions had been first obtained without matter, in which case they describe the pure de Sitter space [15, 16]. Later the matter term was included, [17, 18, 19, 20], first only for special values of the theory parameters, and then in the general case [21, 22]. If the massive gravity theory were indeed the correct theory of gravity, these solutions could describe our universe. However, perturbations around these FLRW backgrounds are expected to be inhomogeneous – due to the non-diagonal metric components, although this effect should be suppressed by the smallness of  $m$  [18]. For this reason solutions of this type are sometimes called in the literature ‘inhomogeneous’, or solutions with ‘inhomogeneous Stückelberg fields’.

For solutions of the two types discussed above the metrics are the same as in GR and the graviton mass manifests itself only as an effective cosmological term. By contrast, the third type of the known bigravity solutions, where the two metrics are bidiagonal in the same frame but not proportional, leads to more complex equations of motion which usually require a numerical analysis. The FLRW solutions can be self-accelerating, but can also show more complex behaviors [23, 24, 25, 26]. There are also anisotropic cosmological solutions, and these show that the generic state to which the universe approaches at late times is anisotropic, and the anisotropy energy behaves similarly to the energy of a non-relativistic matter [27]. The static vacuum solutions with bidiagonal metrics can be rather complex and describe black holes with a massive graviton hair and also gravitating lumps of energy [28]. None of these solutions are asymptotically flat. The asymptotic flatness is achieved by including a regular matter source, which gives rise to solutions which exhibit the Vainshtein mechanism of recovery of General Relativity in a finite region of space [29].

Below, a more detailed description of the currently known bigravity and massive gravity solutions is given.

## 2. Ghost-free bigravity theory

The theory of the ghost-free bigravity [2] is defined on a four-dimensional spacetime manifold equipped with two metrics,  $g_{\mu\nu}$  and  $f_{\mu\nu}$ , which describe two interacting gravitons, one of which is massive and the other one is massless. The kinetic term of each metric is chosen to be of the standard Einstein-Hilbert form, while the interaction between them is described by a local potential term  $\mathcal{U}[g, f]$  which does not contain derivatives and which is expressed by a scalar function of the tensor

$$\gamma^\mu_\nu = \sqrt{g^{\mu\alpha} f_{\alpha\nu}}. \quad (2.1)$$

Here  $g^{\mu\nu}$  is the inverse of  $g_{\mu\nu}$  and the square root is understood in the matrix sense, i.e.

$$(\gamma^2)^\mu_\nu \equiv \gamma^\mu_\alpha \gamma^\alpha_\nu = g^{\mu\alpha} f_{\alpha\nu}. \quad (2.2)$$

Notice that the matrix square root  $\gamma^\mu_\nu$  is well-defined when  $g_{\mu\nu}$  and  $f_{\mu\nu}$  are close to each other so that  $g^{\mu\alpha} f_{\alpha\nu}$  is close to the unit matrix, in which case  $\gamma^\mu_\nu$  is also close to the unit matrix. However,  $\gamma^\mu_\nu$  might cease to be well-defined (and/or real) for generic g,f.

Assuming a g-matter and an f-matter interacting, respectively, only with  $g_{\mu\nu}$  and with  $f_{\mu\nu}$ , the action is (with the metric signature  $- + ++$ )

$$\begin{aligned} S[g, f, \text{matter}] = & \frac{1}{2\kappa_g^2} \int d^4x \sqrt{-g} R(g) + \frac{1}{2\kappa_f^2} \int d^4x \sqrt{-f} \mathcal{R}(f) \\ & - \frac{m^2}{\kappa^2} \int d^4x \sqrt{-g} \mathcal{U}[g, f] + S_g^{[\text{m}]}[g, \text{g-matter}] + S_f^{[\text{m}]}[f, \text{f-matter}], \end{aligned} \quad (2.3)$$

where  $R$  and  $\mathcal{R}$  are the Ricci scalars for  $g_{\mu\nu}$  and  $f_{\mu\nu}$ , respectively,  $\kappa_g^2 = 8\pi G$  and  $\kappa_f^2 = 8\pi G$  are the corresponding gravitational couplings, while  $\kappa^2 = \kappa_g^2 + \kappa_f^2$  and  $m$  is the graviton mass. The interaction between the two metrics is given by

$$\mathcal{U} = \sum_{n=0}^4 b_k \mathcal{U}_k(\gamma), \quad (2.4)$$

where  $b_k$  are parameters, while  $\mathcal{U}_k(\gamma)$  are defined by the relations

$$\begin{aligned} \mathcal{U}_0(\gamma) &= 1, \\ \mathcal{U}_1(\gamma) &= \sum_A \lambda_A = [\gamma], \\ \mathcal{U}_2(\gamma) &= \sum_{A < B} \lambda_A \lambda_B = \frac{1}{2!}([\gamma]^2 - [\gamma^2]), \\ \mathcal{U}_3(\gamma) &= \sum_{A < B < C} \lambda_A \lambda_B \lambda_C = \frac{1}{3!}([\gamma]^3 - 3[\gamma][\gamma^2] + 2[\gamma^3]), \\ \mathcal{U}_4(\gamma) &= \lambda_0 \lambda_1 \lambda_2 \lambda_3 = \frac{1}{4!}([\gamma]^4 - 6[\gamma]^2[\gamma^2] + 8[\gamma][\gamma^3] + 3[\gamma^2]^2 - 6[\gamma^4]). \end{aligned} \quad (2.5)$$

Here  $\lambda_A$  ( $A = 0, 1, 2, 3$ ) are the eigenvalues of  $\gamma^\mu_\nu$ , and, using the hat to denote matrices, one has defined  $[\gamma] = \text{tr}(\hat{\gamma}) \equiv \gamma^\mu_\mu$ ,  $[\gamma^k] = \text{tr}(\hat{\gamma}^k) \equiv (\gamma^k)^\mu_\mu$ . The (real) parameters  $b_k$  could be arbitrary, however, if one requires flat space to be a solution of the theory, and  $m$  to

be the Fierz-Pauli mass of the graviton [3], then the five  $b_k$ 's are expressed in terms of two free parameters  $c_3, c_4$  as follows:

$$\begin{aligned} b_0 &= 4c_3 + c_4 - 6, & b_1 &= 3 - 3c_3 - c_4, & b_2 &= 2c_3 + c_4 - 1, \\ b_3 &= -(c_3 + c_4), & b_4 &= c_4. \end{aligned} \quad (2.6)$$

The theory (2.3) propagates  $7=5+2$  degrees of freedom corresponding to the polarizations of two gravitons, one massive and one massless. Before this theory was discovered [2], other bigravity models, sometimes called f-g theories, had been considered [30]. Such theories propagate 7+1 degrees of freedom, the additional one being the BD ghost [5].

### 2.1. Field equations

Let us introduce the mixing angle  $\eta$  such that  $\kappa_g = \kappa \cos \eta$ ,  $\kappa_f = \kappa \sin \eta$ . Varying the action (2.3) gives the field equations

$$G_\nu^\mu = m^2 \cos^2 \eta T_\nu^\mu + T^{[m]\mu}_\nu, \quad (2.7)$$

$$\mathcal{G}_\nu^\mu = m^2 \sin^2 \eta \mathcal{T}_\nu^\mu + \mathcal{T}^{[m]\mu}_\nu, \quad (2.8)$$

where  $G_\nu^\mu$  and  $\mathcal{G}_\nu^\mu$  are the Einstein tensors for  $g_{\mu\nu}$  and  $f_{\mu\nu}$  and where the two gravitational couplings are included into the definition of the matter sources  $T^{[m]\mu}_\nu$  and  $\mathcal{T}^{[m]\mu}_\nu$ . The “graviton” energy-momentum tensors obtained by varying the interaction  $\mathcal{U}$  are

$$T_\nu^\mu = \tau_\nu^\mu - \mathcal{U} \delta_\nu^\mu, \quad \mathcal{T}_\nu^\mu = -\frac{\sqrt{-g}}{\sqrt{-f}} \tau_\nu^\mu, \quad (2.9)$$

where

$$\begin{aligned} \tau_\nu^\mu &= \{b_1 \mathcal{U}_0 + b_2 \mathcal{U}_1 + b_3 \mathcal{U}_2 + b_4 \mathcal{U}_3\} \gamma_\nu^\mu \\ &\quad - \{b_2 \mathcal{U}_0 + b_3 \mathcal{U}_1 + b_4 \mathcal{U}_2\} (\gamma^2)_\nu^\mu \\ &\quad + \{b_3 \mathcal{U}_0 + b_4 \mathcal{U}_1\} (\gamma^3)_\nu^\mu \\ &\quad - b_4 \mathcal{U}_0 (\gamma^4)_\nu^\mu \end{aligned} \quad (2.10)$$

with  $\mathcal{U}_k \equiv \mathcal{U}_k(\gamma)$ . In deriving these expressions one uses the relations

$$\begin{aligned} \frac{\delta[\gamma^n]}{\delta g^{\mu\nu}} &= \frac{n}{2} g_{\mu\alpha} (\gamma^n)_\nu^\alpha = \frac{n}{2} g_{\nu\alpha} (\gamma^n)_\mu^\alpha, \\ \frac{\delta[\gamma^n]}{\delta f^{\mu\nu}} &= -\frac{n}{2} f_{\mu\alpha} (\gamma^n)_\nu^\alpha = -\frac{n}{2} f_{\nu\alpha} (\gamma^n)_\mu^\alpha, \end{aligned} \quad (2.11)$$

which can be obtained by varying the definition of  $\gamma_\nu^\mu$  and using the properties of the trace. The matter sources are conserved due to the diffeomorphism invariance of the matter terms in the action,  $\overset{(g)}{\nabla}_\mu T^{[m]\mu}_\nu = 0$ ,  $\overset{(f)}{\nabla}_\mu \mathcal{T}^{[m]\mu}_\nu = 0$ , where  $\overset{(g)}{\nabla}$  and  $\overset{(f)}{\nabla}$  are covariant derivatives with respect to  $g_{\mu\nu}$  and  $f_{\mu\nu}$ . The Bianchi identities for (2.7) imply that the graviton energy-momentum tensor is also conserved,  $\overset{(g)}{\nabla}_\mu T_\nu^\mu = 0$ . Similarly, the Bianchi identities imply that  $\overset{(f)}{\nabla}_\mu \mathcal{T}_\nu^\mu = 0$ , but in fact this condition is not independent and

follows from  $\overset{(g)}{\nabla}_\mu T_\nu^\mu = 0$  in view of the diffeomorphism invariance of the interaction term in the action.

If  $\eta \rightarrow 0$  and  $\sin^2 \eta \mathcal{T}_\nu^\mu \rightarrow 0$ , then equations (2.8) for the f-metric decouple and their solution enters the g-equations (2.7) as the fixed reference metric. The g-equations describe in this case a massive gravity theory. For example, if f becomes flat for  $\eta \rightarrow 0$ , then one recovers the dRGT theory [1]. Therefore, theories of massive gravity are contained in the bigravity. However, one has to emphasize that the existence of the massive gravity limit is not guaranteed [31], since  $\sin^2 \eta \mathcal{T}_\nu^\mu$  does not necessarily vanish when  $\eta \rightarrow 0$ , and even if it does, the f-metric does not necessarily reduce to something fixed. For example, in the absence of the f-matter, f-metric becomes Ricci flat in the limit, but this does not guarantee that it is flat. Therefore, although the massive gravity can be embedded into bigravity, it is not a limit of the latter but a different theory with different properties. As a result, the recent observation that the massive gravity theory could be acausal [32] does not necessarily apply to the bigravity theory [33].

### 3. Proportional backgrounds

The simplest solutions of the bigravity equations are obtained by assuming the two metrics to be proportional,

$$f_{\mu\nu} = C^2 g_{\mu\nu}. \quad (3.1)$$

One obtains in this case  $\gamma_\nu^\mu = C \delta_\nu^\mu$  and so

$$\tau_\nu^\mu = (b_1 + 3b_2 C + 3b_3 C^2 + b_4 C^3) C \delta_\nu^\mu, \quad (3.2)$$

which gives the energy-momentum tensors

$$T_\nu^\mu = -\Lambda_g(C) \delta_\nu^\mu, \quad \mathcal{T}_\nu^\mu = -\Lambda_f(C) \delta_\nu^\mu, \quad (3.3)$$

with

$$\begin{aligned} \Lambda_g(C) &= m^2 \cos^2 \eta (b_0 + 3b_1 C + 3b_2 C^2 + b_3 C^3), \\ \Lambda_f(C) &= m^2 \frac{\sin^2 \eta}{C^3} (b_1 + 3b_2 C + 3b_3 C^2 + b_4 C^3). \end{aligned} \quad (3.4)$$

Since the energy-momentum tensors should be conserved, it follows that  $C$  is a constant. As a result, one finds two sets of Einstein equations,

$$G_\mu^\nu + \Lambda_g(C) \delta_\mu^\nu = T_\nu^\mu, \quad \mathcal{G}_\mu^\nu + \Lambda_f(C) \delta_\mu^\nu = \mathcal{T}_\nu^\mu. \quad (3.5)$$

Since one has  $\mathcal{G}_\mu^\nu = G_\mu^\nu/C^2$ , it follows that  $\Lambda_f = \Lambda_g/C^2$ , which gives an algebraic equation for  $C$ , with  $\chi = \tan^2 \eta$ ,

$$(b_0 + 3b_1 C + 3b_2 C^2 + b_3 C^3) = \frac{\chi^2}{C} (b_1 + 3b_2 C + 3b_3 C^2 + b_4 C^3). \quad (3.6)$$

It follows also that the matter sources should be fine-tuned such that  $\mathcal{T}^{[m]\mu}_\nu = T^{[m]\mu}_\nu/C^2$ . Therefore, the independent equations are the same as in GR,

$$G_\mu^\nu + \Lambda_g(C) \delta_\mu^\nu = T_\nu^\mu. \quad (3.7)$$

If the parameters  $b_k$  are chosen according to Eq. (2.6), then Eq.(3.6) factorizes,

$$0 = (C - 1)[(c_3 + c_4)C^3 + (3 - 5c_3 + (\chi - 2)c_4)C^2 + ((4 - 3\chi)c_3 + (1 - 2\chi)c_4 - 6)C + (3c_3 + c_4 - 1)\chi], \quad (3.8)$$

while

$$\frac{\Lambda_g}{m^2 \cos^2 \eta} = (1 - C)((c_3 + C_4)C^2 + (3 - 5c_3 - 2c_4)C + 4c_3 + c_4 - 6).$$

Depending on values of  $c_3, c_4, \eta$ , the equation (3.8) can have up to four real roots. For example, for  $c_3 = 1, c_4 = 0.3, \eta = 1$  the roots are

$$C = \{-2.24; 1; 0.06; 3.61\}, \quad \frac{\Lambda_g(C)}{m^2} = \{10.12; 0; -0.50; -4.50\}. \quad (3.9)$$

As a result, there can be solutions with four different values of the cosmological constant, which can be positive, negative, or zero. If  $C = 1$  then  $\Lambda_g = \Lambda_f = 0$  and the two metrics and the matter sources are identical,  $g_{\mu\nu} = f_{\mu\nu}, T^{[m]\mu}_{\nu} = \mathcal{T}^{[m]\mu}_{\nu}$ . Setting  $T^{[m]\mu}_{\nu} = 0$ , the vacuum GR is recovered, and, in particular, flat space,  $g_{\mu\nu} = f_{\mu\nu} = \eta_{\mu\nu}$ .

Let us consider small fluctuations around flat space,  $g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}, f_{\mu\nu} = \eta_{\mu\nu} + \delta f_{\mu\nu}$ . Inserting into the general equations (2.7),(2.8) and linearizing with respect to the fluctuations, one finds that the linear combinations

$$h_{\mu\nu}^{(m)} = \cos \eta \delta g_{\mu\nu} + \sin \eta \delta f_{\mu\nu}, \quad h_{\mu\nu}^{(0)} = \cos \eta \delta f_{\mu\nu} - \sin \eta \delta g_{\mu\nu} \quad (3.10)$$

fulfill the Fierz-Pauli equations,

$$\square h_{\mu\nu}^{(m)} + \dots = m^2(h_{\mu\nu}^{(m)} - h_{\mu\nu}^{(m)}\eta_{\mu\nu}), \quad \square h_{\mu\nu}^{(0)} + \dots = 0. \quad (3.11)$$

Therefore, one can identify  $h_{\mu\nu}^{(m)}$  and  $h_{\mu\nu}^{(0)}$  with the massive and massless graviton fields, respectively. It seems that one can explicitly identify in this way the massive and massless degrees of freedom only for fluctuations around proportional backgrounds [34]. Moreover, only for proportional backgrounds the null energy condition is fulfilled both in the g and f sectors [35]. Apart from flat space, other solutions with proportional metrics do not admit the massive gravity limit with flat f-metric.

#### 4. FLRW cosmologies with non-bidiagonal metrics

Let us now make a symmetry assumption and choose both metrics to be invariant under spatial SO(3) rotations. Since the theory is invariant under diffeomorphisms, one can choose the spacetime coordinates such that the g-metric is diagonal. However, the f-metric will in general contain an off-diagonal term, so that

$$\begin{aligned} ds_g^2 &= -Q^2 dt^2 + N^2 dr^2 + R^2 d\Omega^2, \\ ds_f^2 &= -(aQdt + cNdr)^2 + (cQdt - bNdr)^2 + u^2 R^2 d\Omega^2. \end{aligned} \quad (4.1)$$

Here  $Q, N, R, a, b, c, u$  depend on  $t, r$  and  $d\Omega^2 = d\vartheta^2 + \sin^2 \vartheta d\varphi^2$ . One can take

$$\gamma^\mu_{\nu} = \sqrt{g^{\mu\alpha} f_{\alpha\nu}} = \begin{pmatrix} a & cN/Q & 0 & 0 \\ -cQ/N & b & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{pmatrix}, \quad (4.2)$$

whose eigenvalues are

$$\lambda_{0,1} = \frac{1}{2} \left( a + b \pm \sqrt{(a - b)^2 - 4c^2} \right), \quad \lambda_2 = \lambda_3 = u. \quad (4.3)$$

Inserting this to (2.5) gives

$$\begin{aligned} \mathcal{U}_1 &= a + b + 2u, & \mathcal{U}_2 &= u(u + 2a + 2b) + ab + c^2, \\ \mathcal{U}_3 &= u(au + bu + 2ab + 2c^2), & \mathcal{U}_4 &= u^2(ab + c^2). \end{aligned} \quad (4.4)$$

Notice that, although the eigenvalues (4.3) can be complex-valued, the  $\mathcal{U}_k$ 's are always real. It is now straightforward to compute the energy-momentum tensors  $T^\mu_\nu$  and  $\mathcal{T}^\mu_\nu$  defined by (2.9),(2.10). In particular, one finds

$$T^0_r = \frac{cN}{Q} [b_1 + 2b_2u + b_3u^2]. \quad (4.5)$$

It will be assumed in what follows that the g-metric is either static or of the FLRW type, in which cases there is no radial energy flux and  $T^0_r$  should be zero. Therefore, either  $c$  should vanish, or the expression in brackets in (4.5) vanishes. The former option will be considered in the next section, while presently let us assume that  $c \neq 0$  and

$$b_1 + 2b_2u + b_3u^2 = 0. \quad (4.6)$$

This yields

$$u = \frac{1}{b_3} \left( -b_2 \pm \sqrt{b_2^2 - b_1b_3} \right). \quad (4.7)$$

Notice that  $u$  was a priori a function of  $t, r$ , but now it is restricted to be a constant. Using this, one finds that  $T^0_0 = T^r_r = -\lambda_g$  and  $\mathcal{T}^0_0 = \mathcal{T}^r_r = -\lambda_f$  where

$$\lambda_g = b_0 + 2b_1u + b_2u^2, \quad \lambda_f = \frac{b_2 + 2b_3u + b_4u^2}{u^2}. \quad (4.8)$$

The conditions  $\overset{(g)}{\nabla}_\rho T_\lambda^\rho = 0$  reduce in this case to the requirement that  $T^0_0 - T_\vartheta^\vartheta$  should vanish. On the other hand, one finds

$$T^0_0 - T_\vartheta^\vartheta = (b_2 + b_3u)[(u - a)(u - b) + c^2], \quad (4.9)$$

and since this has to vanish, either the first or the second factor on the right should be zero. The former case was considered in [17],[23] (see also [19]). However, requiring that  $b_2 + b_3u = 0$  constrains the possible values of the theory parameters  $b_k$  hence solutions obtained in this way are not general. Let us therefore require that [21, 22]

$$(u - a)(u - b) + c^2 = 0. \quad (4.10)$$

Note that in this constraint  $u$  is a constant, but  $a, b$  and  $c$  are still (unknown) functions of  $t, r$ . In view of this, one has  $T^0_0 = T_\vartheta^\vartheta$  and  $\mathcal{T}^0_0 = \mathcal{T}_\vartheta^\vartheta$  hence both energy-momentum tensors are proportional to the unit tensor,  $T^\mu_\nu = -\lambda_g \delta^\mu_\nu$  and  $\mathcal{T}^\mu_\nu = -\lambda_f \delta^\mu_\nu$ . The field equations (2.7) then reduce to

$$\begin{aligned} G_\lambda^\rho + \Lambda_g \delta_\lambda^\rho &= T^{(m)\rho}_\lambda, \\ \mathcal{G}_\lambda^\rho + \Lambda_f \delta_\lambda^\rho &= \mathcal{T}^{(m)\rho}_\lambda, \end{aligned} \quad (4.11)$$

with  $\Lambda_g = m^2 \cos^2 \eta \lambda$  and  $\Lambda_f = m^2 \sin^2 \eta \lambda_f$ . As a result, the two metrics seemingly decouple one from the other, and the graviton mass gives rise to two cosmological terms. Unlike in the case of proportional metrics, no fine tuning between the two matter sources is needed. However, one has to remember that solutions of (4.11) should fulfill the consistency condition (4.10). If the parameters  $b_k$  are chosen according to (2.6) then  $\lambda_g + u^2 \lambda_f = -(u-1)^2$ , therefore, if  $\Lambda_g > 0$  then  $\Lambda_f < 0$ .

Let us consider time-dependent solutions of the FLRW type and assume the g-matter to be a perfect fluid,  $T^{(m)\rho}_{\lambda} = \text{diag}[-\rho_g(t), P_g(t), P_g(t), P_g(t)]$ , while the f-sector can be chosen to be empty,  $\mathcal{T}^{(m)\rho}_{\lambda} = 0$ . Then the solution of (4.11) is chosen to be

$$\begin{aligned} ds_g^2 &= -dt^2 + \mathbf{a}^2(t) \left( \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right), \quad k = 0, \pm 1, \\ ds_f^2 &= -\Delta(U) dT^2 + \frac{dU^2}{\Delta(U)} + U^2 d\Omega^2, \quad \Delta = 1 - \frac{\Lambda_f}{3} U^2, \end{aligned} \quad (4.12)$$

where  $\mathbf{a}$  fulfills the Friedmann equation,

$$3 \frac{\dot{\mathbf{a}}^2 + k}{\mathbf{a}^2} = \Lambda_g + \rho. \quad (4.13)$$

The g-metric describes an expanding FLRW universe containing the matter  $\rho$  and the positive cosmological term  $\Lambda_g$ . It can be of any spatial type – open, closed or flat. At early times, when  $\mathbf{a}$  is small, the matter term  $\rho$  is dominant, while later  $\Lambda_g$  dominates and the universe enters the acceleration phase. The f-metric is the anti-de Sitter one expressed in static coordinates. One has  $\Lambda_f \sim \sin^2 \eta \rightarrow 0$  when  $\eta \rightarrow 0$ , hence the f-metric becomes flat in this limit. Therefore, the solutions apply both in the bigravity theory and in dRGT massive gravity.

#### 4.1. Imposing the consistency condition

The described above effective decoupling of the two metrics has been observed by several authors, in the massive gravity theory [15, 17, 18, 20] and also in the bigravity [23]. However, such a decoupling is only possible if the consistency condition (4.10) is fulfilled, which requires solving a complicated PDE (see Eq.(4.14) below), and this has not always been done. In [15] the solution was obtained for  $\rho_g = 0$  when the g-metric is the de Sitter one expressed in static coordinates, in which case the PDE becomes an ODE. A solution for a more general FLRW g-metric for  $k = 0$  and for special values of  $b_k$  was found in [18], while the general case was considered in [21, 22].

Let us notice that the two metrics in (4.12) are expressed in two different coordinate systems,  $t, r$  and  $T, U$ , whose relation to each other is not yet known. One has  $T = T(t, r)$  and  $U = U(t, r)$ , so that  $dT = \dot{T}dt + T'dr$  and  $dU = \dot{U}dt + U'dr$ . Inserting this into the f-metric in (4.12) and comparing with the f-metric from (4.1), one finds the metric coefficients  $a, b, c$  in (4.1) expressed in terms of the partial derivatives of  $T, U$ . One also finds that  $U = uR = u \mathbf{a} r$ . Inserting the result into (4.10) gives a non-linear PDE,

$$\Delta \left[ \mathbf{a} \sqrt{1-kr^2} (\dot{U}T' - \dot{T}U') - u^2 \mathbf{a}^2 \right]^2 = u^2 \mathbf{a}^2 A_+ A_-, \quad (4.14)$$

with  $A_{\pm} = \mathbf{a}(\Delta\dot{T} \pm \dot{U}) + \sqrt{1 - kr^2}(U' \pm \Delta T')$ . Since  $u$ ,  $\mathbf{a}$ ,  $U$ ,  $\Delta(U)$  are already known, this equation determines  $T(t, r)$ .

Let us consider the  $\eta \rightarrow 0$  limit, when  $\Lambda_f = 0$  and  $\Delta = 1$ . Then one can find exact solutions of (4.14). If  $k = 0$  then

$$T(t, r) = q \int^t \frac{dt}{\dot{\mathbf{a}}} + \left( \frac{u^2}{4q} + qr^2 \right) \mathbf{a}, \quad (4.15)$$

where  $q$  is an integration constant. This solution agrees with the one obtained in [18] for  $c_3 = c_4 = 0$ ,  $u = 3/2$ . For  $k = \pm 1$  one has

$$T(t, r) = \sqrt{q^2 + ku^2} \int^t \sqrt{\dot{\mathbf{a}}^2 + k} dt + q\mathbf{a}\sqrt{1 - kr^2}. \quad (4.16)$$

If  $\Lambda_f \neq 0$  and  $\Delta \neq 1$  then exact solutions of (4.14) are unknown, however, at least when  $\Lambda_f$  is small, solutions can be constructed perturbatively as  $T = T_0 + \sum_{n \geq 1} (-\Lambda_f/3)^n T_n$ . Here  $T_0$  corresponds to zero order expressions (4.15), (4.16), while the corrections  $T_n$  can be obtained by separating the variables with the ansatz  $T_n = \sum_{m=0}^{n+1} f_m(t)(1 - kr^2)^{m/2}$ .

This completes the construction, since all field equations and the consistency condition are fulfilled. Summarizing, one obtains self-accelerating FLRW solutions of all spatial types, equally valid in the bigravity and massive gravity theories. In the latter case this exhausts all possible homogeneous and isotropic cosmologies. Somewhat confusingly, these solutions are sometimes called in the literature inhomogeneous, because the two metrics are non-bidiagonal and  $T, U$ , which play the role of Stückelberg scalars in the massive gravity limit, are inhomogeneous, as they both depend on  $r$ . It is then expected that fluctuations around the FLRW backgrounds should show an inhomogeneous spectrum, although this effect will be proportional to  $m^2$  and so will be small [18].

In the spatially open case,  $k = -1$ , choosing in (4.16)  $q = u$ , yields  $T = u\mathbf{a}\sqrt{1 + r^2}$ . Inserting this to (4.12), together with  $U = u\mathbf{a}r$ , gives the flat f-metric which turns out to be diagonal *both* in the  $T, U$  and  $t, r$  coordinates,

$$ds_f^2 = -dT^2 + dU^2 + U^2 d\Omega^2 = u^2 \mathbf{a}^2 \left( -\frac{\dot{\mathbf{a}}^2}{\mathbf{a}^2} dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\Omega^2 \right), \quad (4.17)$$

so that the g and f metrics happen to be bidiagonal in this case. This particular solution was found in [36]. It is unclear if it should be physically distinguished when compared to the other solutions in (4.15), (4.16) for generic  $k, q$ . The analysis of [37] shows that this solution is unstable at the nonlinear level, while the case of generic  $k, q$  has not yet been studied.

## 5. FLRW cosmologies with bidiagonal metrics

All solutions considered above are described by the same metrics as in GR, and the graviton mass manifests itself only as the effective cosmological term(s). More general solutions are obtained assuming that both metrics are simultaneously diagonal

[23, 24, 25, 38, 26],

$$\begin{aligned} ds_g^2 &= -dt^2 + \mathbf{a}(t)^2 \left( \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right), \\ ds_f^2 &= -\mathcal{A}(t)^2 dt^2 + \mathbf{b}(t)^2 \left( \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right). \end{aligned} \quad (5.1)$$

The  $G_0^0$  and  $\mathcal{G}_0^0$  equations (2.7), (2.8) then read

$$\frac{\dot{\mathbf{a}}^2}{\mathbf{a}^2} = \frac{\Lambda_g(\xi) + \rho_g}{3} - \frac{k}{\mathbf{a}^2}, \quad \frac{1}{\mathcal{A}^2} \frac{\dot{\mathbf{b}}^2}{\mathbf{b}^2} = \frac{\Lambda_f(\xi) + \rho_f}{3} - \frac{k}{\mathbf{b}^2}, \quad (5.2)$$

where  $\xi = \mathbf{b}/\mathbf{a}$  and where  $\Lambda_g, \Lambda_f$  are defined in (3.4). The  $\overset{(g)}{\nabla}_\mu T_\nu^\mu = 0$  conditions reduce to

$$[\dot{\mathbf{b}} - \mathcal{A}\dot{\mathbf{a}}] (b_1 + 2b_2\xi + b_3\xi^2) = 0. \quad (5.3)$$

### 5.1. Generic solutions

Let us assume that the first factor in (5.3) vanishes,  $\dot{\mathbf{b}} - \mathcal{A}\dot{\mathbf{a}} = 0$ . Comparing this to (5.2) gives the algebraic relation  $\Lambda_g(\xi) + \rho_g = \xi^2(\Lambda_f(\xi) + \rho_f)$ , or, setting for simplicity  $\rho_f = 0$ ,

$$\frac{b_3}{3} \xi^4 + (b_2 - \frac{\chi}{3} b_4) \xi^3 + (b_1 - \xi b_3) \xi^2 + (\frac{b_0}{3} - \chi b_2 + \frac{\rho_g}{3 \cos^2 \eta}) \xi = \frac{\xi b_1}{3}, \quad (5.4)$$

with  $\chi = \tan^2 \eta$ . Assuming the equation of state  $P_g = w_g \rho_g$  one has  $\rho_g = \rho_g^0 \mathbf{a}^{-3(1+w_g)}$ , and so that Eq.(5.4) gives an algebraic relation between  $\mathbf{a}$  and  $\xi$ , whose solution is  $\xi(\mathbf{a})$ . Inserting  $\xi(\mathbf{a})$  into the first equation in (5.2) gives the Friedmann equation,

$$\dot{\mathbf{a}}^2 + V(\mathbf{a}) = -k, \quad V(\mathbf{a}) = -\frac{\mathbf{a}^2}{3}(\Lambda_g(\xi) + \rho_g), \quad (5.5)$$

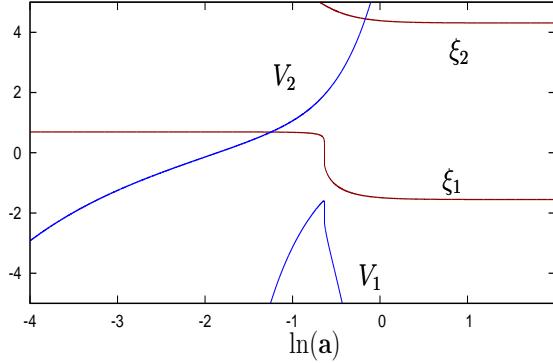
where  $\xi = \xi(\mathbf{a})$  and  $\rho_g = \rho_g(\mathbf{a})$ , hence  $\mathbf{a}$  is confined to the region where  $V(\mathbf{a}) \leq -k$ .

Depending on choice of the solution of (5.4), the shape of the potential  $V(\mathbf{a})$  can be different. The simplest solutions are obtained for  $b_1 = b_3 = 0$  [24]. Making the parameter choice (2.6), the self-accelerating solutions are found only if  $b_1 = 3 - 3c_3 - c_4 < 0$  [23]. In this case Eq.(5.4) has only two real roots  $\xi_1(\mathbf{a})$  and  $\xi_2(\mathbf{a})$  which exist for any positive  $\mathbf{a}$ ,  $\Lambda_g(\xi_1)$  approaching a positive value at large  $\mathbf{a}$  while  $\Lambda_g(\xi_2)$  being negative. This corresponds, respectively, to the accelerating and recollapsing solutions (see Fig.1). For  $b_1 > 0$  there could be more complex solutions for which the universe size oscillates around a finite value [23]. When Eq.(5.5) is solved, the f-metric coefficients are  $\mathbf{b} = \mathbf{a}\xi(\mathbf{a})$  and  $\mathcal{A} = \dot{\mathbf{b}}/\dot{\mathbf{a}}$ .

The perturbative analysis reveals that the self-accelerating solutions are stable [39, 40, 41], hence they can be used for interpreting observational data [42].

### 5.2. Special solutions

Let us now assume that the second factor in (5.3) vanishes,  $b_1 + 2b_2\xi + b_3\xi^2 = 0$ . Notice that this is the same equation as in (4.6). It follows that  $\xi = \mathbf{b}/\mathbf{a}$  is constant, therefore



**Figure 1.** The two solutions  $\xi_1, \xi_2$  of (5.4) and the corresponding effective potentials  $V(a)$  for the same parameter values as in (3.9). One has  $V_1 < -1$ , hence for any  $k = 0, \pm 1$  solutions of (5.5) travel over the  $V_1$  potential barrier and enter the accelerated phase for large  $a$ . Since  $\Lambda_g(\xi_2) < 0$ ,  $V_2 \rightarrow +\infty$  for large  $a$  and solutions of (5.5) bounce from the potential barrier and recollapse.

$\dot{\mathbf{a}}/\mathbf{a} = \dot{\mathbf{b}}/\mathbf{b}$ . Equations then again reduce to (5.5), where  $\Lambda_g$  is now constant. Combining the two equations (5.2) gives

$$\mathcal{A}^2 = \frac{\xi^2 \dot{\mathbf{a}}^2}{\xi^2 \mathbf{a}^2 (\Lambda_f + \rho_f)/3 - k}, \quad (5.6)$$

and this expression can be positive only in special cases. In the massive gravity limit, where  $\rho_f = \eta = \Lambda_f = 0$ ,  $\mathcal{A}^2$  is positive only for  $k = -1$ , which reproduces the solution (4.17), and  $\mathcal{A}$  does not exist for  $k = 0, 1$ , hence there are no FLRW cosmologies with bidiagonal metrics in these cases [18].

## 6. Anisotropic cosmologies

Solutions considered in the previous section can be generalized to the anisotropic case by choosing the two metrics to be bidiagonal and of the same Bianchi type [27],

$$ds_g^2 = -\alpha^2 dt^2 + h_{ab} \omega^a \otimes \omega^b, \quad ds_f^2 = -\mathcal{A}^2 dt^2 + \mathcal{H}_{ab} \omega^a \otimes \omega^b. \quad (6.1)$$

Here  $\alpha, h_{ab}, \mathcal{A}, \mathcal{H}_{ab}$  depend on  $t$ , and the vectors  $e_a$  dual to the one-forms  $\omega^a$  generate a three-parameter translation group acting on the 3-space,  $[e_a, e_b] = C^c_{ab} e_c$ . The structure coefficients are  $C^c_{ab} = n^{cd} \epsilon_{dab}$  with  $n^{ab} = \text{diag}[n^{(1)}, n^{(2)}, n^{(3)}]$ , where  $n^{(a)}$  assume the values shown in Table 1.

This choice of the structure coefficients corresponds to the Bianchi types of class A, in which case one can choose  $h_{ab} = \text{diag}[\alpha_1^2, \alpha_2^2, \alpha_3^2]$  and  $\mathcal{H}_{ab} = \text{diag}[\mathcal{A}_1^2, \mathcal{A}_2^2, \mathcal{A}_3^2]$ , and this guarantees that  $G_r^0 = \mathcal{G}_r^0 = 0$ , so that radial energy fluxes are absent. Choosing

$$\begin{aligned} [\alpha_1, \alpha_2, \alpha_3] &= e^{\Omega} [e^{\beta_+ + \sqrt{3}\beta_-}, e^{\beta_+ - \sqrt{3}\beta_-}, e^{-2\beta_+}], \\ [\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3] &= e^{\mathcal{W}} [e^{\mathcal{B}_+ + \sqrt{3}\mathcal{B}_-}, e^{\mathcal{B}_+ - \sqrt{3}\mathcal{B}_-}, e^{-2\mathcal{B}_+}], \end{aligned} \quad (6.2)$$

the spatial 3-curvature of the g-metric is

$$R^{(3)} = \frac{2n^{(1)}n^{(2)}}{\alpha_3^2} - \frac{1}{2\alpha_1^2\alpha_2^2\alpha_3^2} (n^{(1)}\alpha_1^2 + n^{(2)}\alpha_2^2 - n^{(3)}\alpha_3^2)^2, \quad (6.3)$$

|           | I | II | VI <sub>0</sub> | VII <sub>0</sub> | VIII | IX |
|-----------|---|----|-----------------|------------------|------|----|
| $n^{(1)}$ | 0 | 1  | 1               | 1                | 1    | 1  |
| $n^{(2)}$ | 0 | 0  | -1              | 1                | 1    | 1  |
| $n^{(3)}$ | 0 | 0  | 0               | 0                | -1   | 1  |

**Table 1.** Values of  $n^{(a)}$  for the Bianchi class A types

while the 3-curvature  $\overset{(3)}{\mathcal{R}}$  of the f-metric is obtained from this by replacing  $\alpha_a \rightarrow \mathcal{A}_a$  and  $\Omega \rightarrow \mathcal{W}$ .

The potential (2.4) is  $\mathcal{U}\sqrt{-g} = \alpha U_g + \mathcal{A}U_f$  where

$$U_g = b_0 e^{3\Omega} + b_3 e^{3\mathcal{W}} + b_1 e^{\mathcal{W}+2\Omega} \left( e^{-2(\mathcal{B}_+-\beta_+)} + 2e^{\mathcal{B}_+-\beta_+} \cosh[\sqrt{3}(\mathcal{B}_--\beta_-)] \right) + b_2 e^{2\mathcal{W}+\Omega} \left( e^{2(\mathcal{B}_+-\beta_+)} + 2e^{-(\mathcal{B}_+-\beta_+)} \cosh[\sqrt{3}(\mathcal{B}_--\beta_-)] \right), \quad (6.4)$$

while  $U_f$  is obtained from this by replacing  $b_k \rightarrow b_{k+1}$ . The equations for the g-metric read

$$\begin{aligned} \left( \frac{\dot{\Omega}}{\alpha} \right)^2 &= \left( \frac{\dot{\beta}_+}{\alpha} \right)^2 + \left( \frac{\dot{\beta}_-}{\alpha} \right)^2 + \frac{1}{6} \left[ 2 \cos^2 \eta e^{-3\Omega} U_g - \overset{(3)}{R} + 2\rho_g \right], \\ \left( e^{3\Omega} \frac{\dot{\Omega}}{\alpha} \right)^. &= \frac{1}{6} \left[ \cos^2 \eta \left( \frac{\partial U}{\partial \Omega} + 3\alpha \mathcal{U}_g \right) - 2\alpha e^{3\Omega} \overset{(3)}{R} + 3\alpha e^{3\Omega} (\rho_g - P_g) \right], \\ \left( e^{3\Omega} \frac{\dot{\beta}_\pm}{\alpha} \right)^. &= -\frac{1}{12} \frac{\partial}{\partial \beta_\pm} \left( 2 \cos^2 \eta U - \alpha e^{3\Omega} \overset{(3)}{R} \right), \end{aligned} \quad (6.5)$$

while the equations for the f-metric are obtained from these by replacing  $\Omega \rightarrow \mathcal{W}$ ,  $\beta_\pm \rightarrow \mathcal{W}_\pm$ ,  $\overset{(3)}{R} \rightarrow \overset{(3)}{\mathcal{R}}$ ,  $\mathcal{U}_g \rightarrow \mathcal{U}_f$  and  $\cos \eta \rightarrow \sin \eta$ . The conditions  $\overset{(g)}{\nabla}_\mu T_\nu^\mu = 0$  reduce to

$$\alpha \left( \dot{W} \frac{\partial}{\partial \mathcal{W}} + \dot{\mathcal{B}}_+ \frac{\partial}{\partial \mathcal{B}_+} + \dot{\mathcal{B}}_- \frac{\partial}{\partial \mathcal{B}_-} \right) U_g = \mathcal{A} \left( \dot{\Omega} \frac{\partial}{\partial \Omega} + \dot{\beta}_+ \frac{\partial}{\partial \beta_+} + \dot{\beta}_- \frac{\partial}{\partial \beta_-} \right) \mathcal{U}_f.$$

All quantities in these equations are dimensionless, assuming  $1/m$  to be the length scale, while the energy density is measured in units of  $m^2 M_{\text{pl}}^2$ .

For the Bianchi types I or IX one can set anisotropies to zero,  $\beta_\pm = \mathcal{B}_\pm = 0$ , and then, with  $\mathbf{a} = 2e^\Omega$  and  $\mathbf{b} = 2e^{\mathcal{W}}$ , the above equations reduce to the FLRW equations (5.2),(5.3) for  $k = 0, 1$  (the spatially open  $k = -1$  FLRW case is contained in the Bianchi V type, which does not belong to the class A).

In the Bianchi I case it is consistent to set  $\beta_\pm = \mathcal{B}_\pm$  and then the two metrics are proportional as described in section 3,  $f_{\mu\nu} = C^2 g_{\mu\nu}$ , if only the matter sources are fine-tuned such that  $\rho_f = \rho_g/C^2$ . One has

$$ds_g^2 = -dt^2 + e^{2\Omega} \left( e^{2\beta_+ + \sqrt{3}\beta_-} dx_1^2 + e^{2\beta_+ - \sqrt{3}\beta_-} dx_2^2 + e^{-4\beta_+} dx_3^2 \right), \quad (6.6)$$

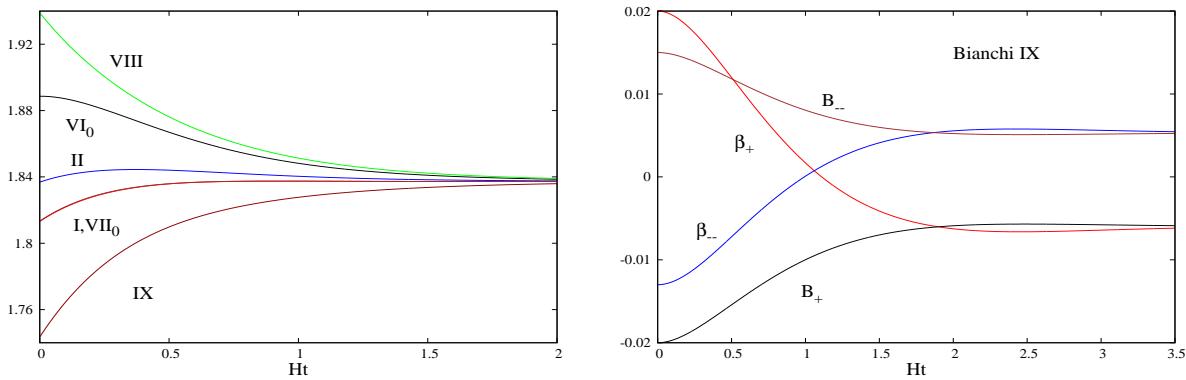
where  $\Omega, \beta_{\pm}$  fulfill (assuming that  $\alpha = 1$ )

$$\begin{aligned}\dot{\Omega}^2 &= (\sigma_+^2 + \sigma_-^2) e^{-6\Omega} + \frac{1}{3} (\Lambda_g(C) + \rho_g), \\ \dot{\beta}_{\pm} &= \sigma_{\pm} e^{-3\Omega},\end{aligned}\quad (6.7)$$

with constant  $\sigma_{\pm}$ . Here  $C, \Lambda_g(C) \equiv 3H^2$  are defined by (3.4),(3.6). At late times solutions of Eqs.(6.7) approach configurations with constant anisotropies,  $\Omega = Ht + O(e^{-3Ht}), \beta_{\pm} = \beta_{\pm}(\infty) + O(e^{-3Ht})$ .

It turns out that solutions for all other Bianchi types and for non fine-tuned sources also evolve into configurations with equal and nonvanishing anisotropies,  $\beta_{\pm} = \mathcal{B}_{\pm}$ . This can be seen by applying a numerical procedure [27], whose input configuration at the initial time moment  $t = 0$  is an anisotropic deformation of a finite size FLRW universe. The initial value of  $\Omega$  is zero, hence the initial universe size  $e^{\Omega} \sim 1$  (in 1/m units), while the initial anisotropies are  $\beta_{\pm}, \mathcal{B}_{\pm}, \dot{\beta}_{\pm}, \dot{\mathcal{B}}_{\pm} \sim 10^{-2}$ . The initial values of  $\dot{\Omega}, \mathcal{W}, \dot{\mathcal{W}}$  are determined by resolving the first order constraints contained in the field equations. It is assumed that the f-sector is empty,  $\rho_f = 0$ , whereas the g-sector contains a radiation and a non-relativistic matter,  $\rho_g = 0.25 \times e^{-4\Omega} + 0.25 \times e^{-3\Omega}$ , so that the dimensionful energy  $m^2 M_{\text{pl}}^2 \rho_g \sim 10^{-10}(\text{eV})^4$ , assuming that  $m \sim 10^{-33}\text{eV}$ .

The numerical extension to  $t > 0$  of the  $t = 0$  initial data reveals that, for all Bianchi types, the universe approaches the state with proportional metrics in which the expansion rate  $\dot{\Omega}$  is constant,  $H$ , and the anisotropies  $\beta_{\pm} = \mathcal{B}_{\pm}$  are also constant. For the Bianchi I solutions the constant anisotropies can be absorbed by redefining the spatial coordinates, but not for other Bianchi types. This means that the universe generically runs into anisotropic states at late times. These properties of the solutions



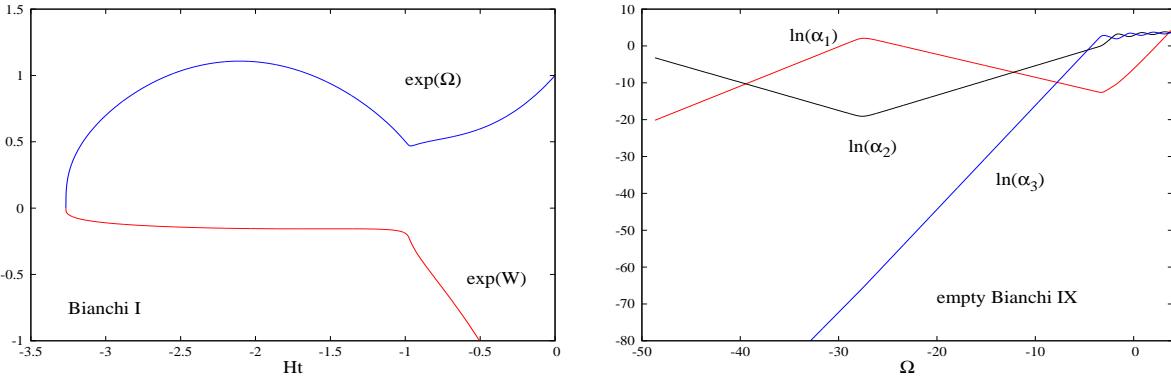
**Figure 2.** The Hubble parameter  $\dot{\Omega}$  for all Bianchi types of class A (left) and the anisotropy amplitude for the Bianchi IX solution (right).

are illustrated in Fig.2. The relative shear contribution to the total energy density  $\Sigma = \sqrt{\dot{\beta}_+^2 + \dot{\beta}_-^2}/\dot{\Omega}$  tends to zero at late times, however, if only just one or two Hubble times have elapsed since the beginning of the current phase of the universe acceleration,  $\Sigma$  should not necessarily be small. It turns out that at late times the anisotropies oscillate around their asymptotic values. Linearizing the field equations with respect to

small deviations from the proportional background  $f_{\mu\nu} = C^2 g_{\mu\nu}$  to which the solutions approach, one finds that

$$\dot{\beta}_{\pm} \sim \dot{\mathcal{B}}_{\pm} \sim e^{-3Ht/2} \cos(H\omega t), \quad (6.8)$$

where  $\omega$  can be expressed in terms of  $C, b_k, \eta$  [27]. It follows that the shear energy is  $\dot{\beta}_+^2 + \dot{\beta}_-^2 \sim e^{-3\Omega} \sim 1/a^3$ , while in GR it decreases as  $\sim 1/a^6$ . The fact that the shear energy in the bigravity theory shows the same fall-off rate as a cold dark matter suggests that the latter could in fact be an anisotropy effect, although it is unclear if this interpretation can also explain the dark matter clustering.



**Figure 3.** Near singularity behavior for the Bianchi I (left) and Bianchi IX (right) solutions.

When continued to the past, the solutions hit a singularity where both  $e^{\Omega}$  and  $e^W$  vanish (see Fig.3). The Bianchi IX solutions, when approaching singularity, show the typical billiard behavior characterized by a sequence of Kasner-type periods [43]. During each period one has  $\alpha_p \propto t^{p_a}$  where  $p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1$ , such that two of the three  $\alpha_a$ 's grow in time whereas the third one decreases, until the next period starts when one of the growing amplitudes becomes decreasing (see Fig.3).

## 7. Black holes, lumps, stars and the Vainshtein mechanism

Historically, the first bigravity black holes were obtained in the generic f-g bigravity theory [30] for non-bidiagonal metrics [44]. Their counterparts in the ghost-free bigravity under consideration are described by the ansatz (4.1). The field equations then reduce to (4.11) (without matter terms), and the solution is the Schwarzschild-de Sitter metric,

$$ds_g^2 = -Ddt^2 + \frac{dr^2}{D} + r^2 d\Omega^2, \quad D = 1 - \frac{2M}{r} - \frac{\Lambda_g r^2}{3}, \quad (7.1)$$

with  $\Lambda_g$  defined after Eq.(4.11). The f-metric is still expressed in terms of  $T, U$  by (4.12), and to fulfill the consistency condition (4.10) the procedure is the same as in Section 4, which gives  $U = ur$  and  $T = ut - u \int \frac{D-\Delta}{D\Delta} dr$  with  $u$  from (4.7) [28]. Since the f-metric becomes flat for  $\eta \rightarrow 0$ , this solution describes black holes also in the dRGT massive gravity [15, 16] (see also [47]).

These solutions and their generalizations for a nonzero electric charge [45] exhaust all known black holes in the massive gravity theory. In particular, there are no asymptotically flat black holes in this case (see [46] for a recent discussion). However, in the bigravity one finds more solutions when the metrics are bidiagonal [28],

$$ds_g^2 = Q^2 dt^2 - \frac{dr^2}{N^2} - r^2 d\Omega^2, \quad ds_f^2 = A^2 dt^2 - \frac{U'^2}{Y^2} dr^2 - U^2 d\Omega^2. \quad (7.2)$$

Here  $Q, N, Y, U, a$  are 5 functions of  $r$  which fulfill the equations

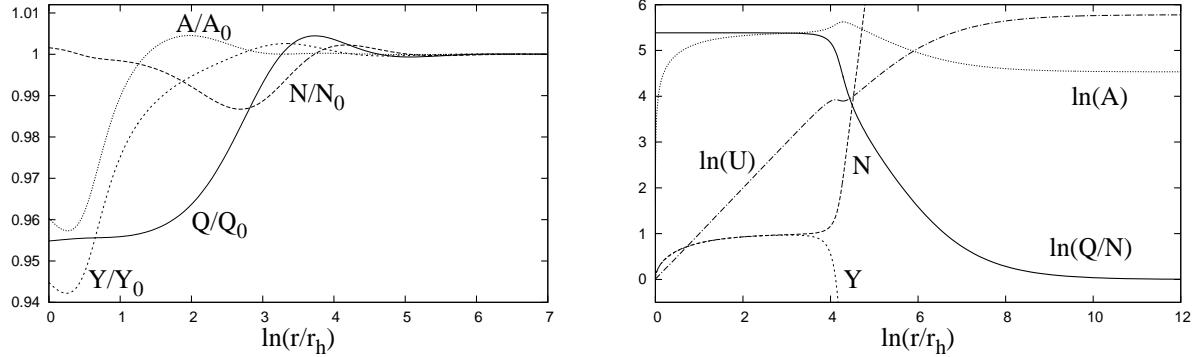
$$\begin{aligned} G_0^0 &= m^2 \cos^2 \eta T_0^0, & G_r^r &= m^2 \cos^2 \eta T_r^r, \\ \mathcal{G}_0^0 &= m^2 \sin^2 \eta \mathcal{T}_0^0, & \mathcal{G}_r^r &= m^2 \sin^2 \eta \mathcal{T}_r^r, \\ T_r^{r'} + \frac{Q'}{Q} (T_r^r - T_0^0) + \frac{2}{r} (T_\vartheta^\vartheta - T_r^r) &= 0. \end{aligned} \quad (7.3)$$

The simplest solutions are again obtained for the proportional metrics,  $f_{\mu\nu} = C^2 g_{\mu\nu}$ , with  $g_{\mu\nu}$  given by (7.1) where  $\Lambda_g = \Lambda_g(C)$  is defined by (3.4),(3.6). Since  $\Lambda_g$  can be positive, negative, or zero, there are the Schwarzschild (S), Schwarzschild-de Sitter (SdS), and Schwarzschild-anti-de Sitter (SAdS) black holes. Let us call them background black holes. More general solutions are obtained by numerically integrating Eqs.(7.3).

In doing this, it is assumed that the g-metric has a regular event horizon at  $r = r_h$ , where  $T_\nu^\mu$  and the curvature are finite. A detailed analysis shows [28] that the horizon should be common for both metrics, therefore amplitudes  $Q^2, N^2, A^2, Y^2$  should have a simple zero, while  $U, U'$  should be non-zero at  $r = r_h$ . Next, one finds that the boundary conditions at the horizon comprise a one-parameter family labeled by  $u = U(r_h)/r_h$ , the ratio of the event horizon radius measured by the f-metric to that measured by the g-metric. Finally, it turns out that the horizon surface gravities and temperatures determined with respect to both metrics are the same [48].

Choosing a value of  $u$  and integrating numerically the equations from  $r = r_h$  towards large  $r$ , the result is as follows [28]. If  $u = C$  where  $C$  fulfills Eq.(3.6) then the solution is one of the background black holes described above. If  $u = C + \delta u$  with a small  $\delta u$ , then one can expect the solution to be the background black hole slightly deformed by a massive graviton ‘hair’ localized in the horizon vicinity. This is confirmed for the SAdS solutions ( $\Lambda_g < 0$ ), which indeed support a short massive hair and show deviations from the pure SAdS in the horizon vicinity, but far away from the horizon the deviations tend to zero (see Fig.4).

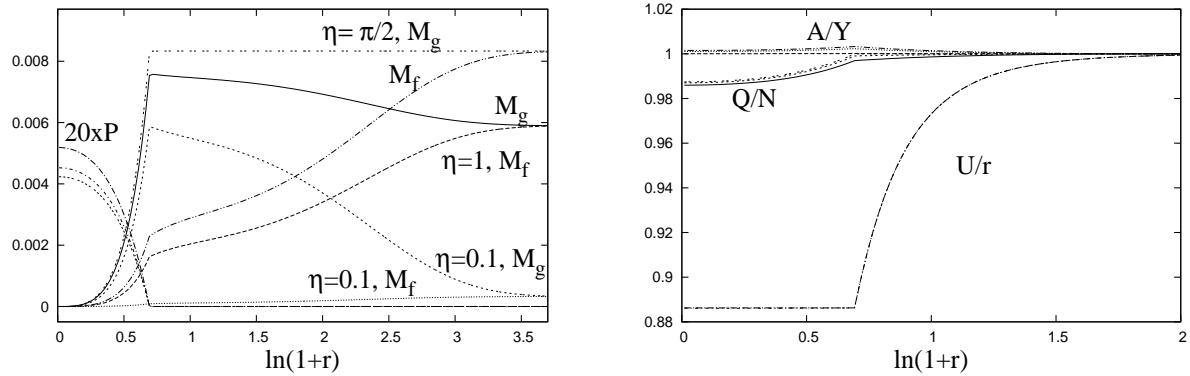
However, the argument does not work for  $\Lambda_g \geq 0$ . When one deforms the S background by setting  $u = r_h + \delta u$  for however small  $\delta u$ , the solutions first stay very close to Schwarzschild (until  $\ln(r/r_h) \sim 4$  in Fig.4). However, for large  $r$  they deviate from the background and show a completely different asymptotic behavior at infinity, characterized by a quasi-AdS g-metric and a compact f-metric giving a finite value for the volume of the 3-space [28]. Therefore, the only asymptotically flat black hole in the bigravity theory is pure Schwarzschild, and deforming it results in losing the asymptotic flatness. Similarly, trying to deform the SdS background produces a curvature singularity at a finite proper distance away from the black hole horizon, hence the only asymptotically de Sitter black hole is the pure SdS.



**Figure 4.** Hairy deformations of the SAdS background, where  $A_0, N_0, Q_0, Y_0$  correspond to the undeformed solution (left); and of the S background (right).

In the shrinking horizon limit,  $r_h \rightarrow 0$ , the black hole ‘hair’ does not disappear but becomes a static ‘lump’ made of massive field modes. Such lumps are described by globally regular solutions for which the event horizon is replaced by the regular center at  $r = 0$ , while at infinity the asymptotic behavior is the same as for the black holes [28]. None of the lumps are asymptotically flat (apart from the flat space).

Asymptotically flat solutions can be obtained by adding matter. Suppose that the f-sector is empty, while the g-sector contains  $T^{[m]\mu}_{\nu} = \text{diag}[-\rho(r), P(r), P(r), P(r)]$  with  $\rho = \rho_\star \theta(r_\star - r)$ , corresponding to a ‘star’ with a constant density  $\rho_\star$  and a radius  $r_\star$ . Adding this source to the field equations (7.3), assuming the regular center at  $r = 0$  and integrating towards large  $r$ , one finds a solution for which both metrics approach Minkowski metric at infinity. Introducing the mass functions  $M_g, M_f$  via  $g^{rr} = N^2 = 1 - 2M_g(r)/r$  and  $f^{rr} = Y^2/U^2 = 1 - 2M_f(r)/r$ , one finds that  $M_g, M_f$  rapidly increase inside the star, while outside they approach the same asymptotic value  $M_g(\infty) = M_f(\infty) \sim \sin^2 \eta$  (see Fig.5). For  $\eta = \pi/2$  the g-metric is coupled only to



**Figure 5.** Profiles of the asymptotically flat star solution sourced by a regular matter distribution.

the matter and is described by the Schwarzschild solution,  $M_g(r) = \rho_\star r^3/6$  for  $r < r_\star$  and  $M_g(r) = \rho_\star r_\star^3/6 \equiv M_{\text{ADM}}$  for  $r > r_\star$ . For  $\eta < \pi/2$  the star mass  $M_{\text{ADM}}$  is partially

screened by the negative graviton energy. For  $\eta = 0$  the f-metric becomes flat, so that  $M_f = 0$ , while  $M_g$  asymptotically approaches zero and the star mass is totally screened, because the massless graviton decouples and there could be no  $1/r$  terms in the metric.

If the graviton mass is very small, then the  $m^2 T_\nu^\mu$  contribution to the equations is small as compared to  $T^{[m]\mu}_\nu$ , and for this reason  $M_g$  rests approximately constant for  $r_\star < r < r_V \sim (M_{\text{ADM}}/m^2)^{1/3}$ . This illustrates the Vainshtein mechanism of recovery of General Relativity in a finite region [29]. This mechanism has also been independently confirmed within the generic massive gravity theory with the BD ghost [49, 50], and in the dRGT theory [51]. It is worth noting that the mechanism works only in the presence of a regular matter source, and not for vacuum black holes or lumps.

To recapitulate, globally regular and asymptotically flat stars exist both in the bigravity theory and in the dRGT massive gravity. In the latter case there are also SdS black holes, but there are no asymptotically flat black holes. There are hairy black holes in the bigravity theory, but the only one which is asymptotically flat is the standard “hairless” Schwarzschild black hole.

## 8. Concluding remarks

Summarizing what was said above, all known cosmologies and black holes in the dRGT massive gravity theory are obtained within the inhomogeneous Stückelberg field approach, assuming non-bidiagonal metrics [36]. Perturbations of the cosmological solutions were studied in [52, 53, 54, 55]. One could perhaps also mention a peculiar FLRW solution [17] with a *degenerate* f-metric, although its g-metric is regular and is sourced by the effective  $T_0^0 = \sum_{n=0}^3 a_n \mathbf{a}^{-n}$  with constant  $a_n$  (see also [56, 57]).

Although massive gravity solutions with bidiagonal metrics similar to (4.17) are very special and hence are unlikely to be physically important, they are popular due to their simplicity. They have been studied also in the modified versions of the theory, as for example in the extended massive gravity [58, 59, 60] where the graviton mass  $m$  is promoted to a dynamical field [18, 61], or in the massive gravity theory with a (quasi) dilaton field [62, 63]. Choosing the fixed f-metric to be non-flat, as for example de Sitter or FLRW, also leads to new cosmological [64, 65, 66, 67, 68, 69] and black holes solutions [70], and allows one to study [71, 72, 73] the Higuchi bound [74], that is the limit where the scalar polarization of the massive graviton decouples. In addition, O(4)-symmetric instanton solutions in the Euclidean version of the massive gravity theories have been considered as well [75, 76].

Changing the reference metric from case to case does not seem to be natural. If one wants to consider different possibilities for f, then it is logical to make it dynamical, which leads to the bigravity. The massive gravities with a fixed f then can be recovered by choosing a source for f, for example a constant  $\rho_f = P_f > 0$ , and taking the limit  $\eta \rightarrow 0$ . For  $\eta \neq 0$  both metrics are dynamical, which gives rise to a much richer solution structure than in the massive gravity theories.

## Acknowledgments

It is a pleasure to acknowledge discussions with Thibault Damour and thank him for a careful reading of the manuscript and valuable remarks. I would like also to thank the IHES for hospitality during my visit in March 2013 when this article was written, and I thank Shinji Mukohyama for inviting me to write it for Classical and Quantum Gravity.

## References

- [1] de Rham C and Gabadadze G and Tolley A G 2011 *Phys.Rev.Lett.* **106** 231101
- [2] Hassan S F and Rosen R A 2012 *JHEP* **1202** 126
- [3] Fierz M and Pauli W 1939 *Proc.Roy.Soc.Lond.* **A173** 211
- [4] Hinterbichler K 2012 *Rev.Mod.Phys.* **84** 671
- [5] Boulware D G and Deser S 1972 *Phys.Rev.* **D6** 3368
- [6] Hassan S F and Rosen R A 2012 *Phys.Rev.Lett.* **108** 041101
- [7] Golovnev A 2012 *Phys.Lett.* **B707** 404
- [8] Kluson J 2012 *Phys.Rev.* **D86** 044024
- [9] Hassan S F and Schmidt-May A and von Strauss M 2012 *Phys.Lett.* **B715** 335
- [10] Hassan S F and Rosen R A 2012 *JHEP* **1204** 123
- [11] Deffayet C and Mourad J and Zahariade G 2013 *JCAP* **1301** 032
- [12] Riess A G et al. 1998 *Astron.Journ.* **116** 1009
- [13] Perlmutter S et al. 1999 *Astrophys.Journ.* **517** 565
- [14] Damour T and Kogan I I and Papazoglou A 2002 *Phys.Rev.* **D66** 104025
- [15] Koyama K and Niz G and Tasinato G 2011 *Phys.Rev.Lett.* **107** 131101
- [16] Koyama K and Niz G and Tasinato G 2011 *Phys.Rev.* **D84** 064033
- [17] Chamseddine A H and Volkov M S 2011 *Phys.Lett.* **B704** 652
- [18] D'Amico G and de Rham C and Dubovsky S and Gabadadze G and Pirtskhalava D and Tolley A J 2011 *Phys.Rev.* **D84** 124046
- [19] Kobayashi T and Siino M Yamaguchi M and Yoshida D 2012 *Phys.Rev.* **D86** 061505
- [20] Gratia P and Hu W and Wyman M 2012 *Phys.Rev.* **D86** 061504
- [21] Volkov M S 2012 *Phys.Rev.* **D86** 061502
- [22] Volkov M S 2012 *Phys.Rev.* **D86** 104022
- [23] Volkov M S 2012 *JHEP* **1201** 035
- [24] von Strauss M and Schmidt-May A and Enander J and Mortsell E and Hassan S F 2012 *JCAP* **1203** 042
- [25] Comelli D and Crisostomi M and Nesti F and Pilo L 2012 *JHEP* **1203** 067
- [26] Capozziello S and Martin-Moruno P 2012 *Phys.Lett.* **B719** 14
- [27] Maeda K and Volkov M S 2013 Anisotropic universes in the ghost-free bigravity. *Preprint* arXiv:1302.6198
- [28] Volkov M S 2012 *Phys.Rev.* **D85** 124043
- [29] Vainshtein A I 1972 *Phys.Lett.* **B39** 393
- [30] Isham C J and Salam A and Strathdee J A 1971 *Phys.Rev.* **D3** 867
- [31] Baccetti V and Martin-Moruno P and Visser M 2013 *Class.Quant.Grav.* **30** 015004
- [32] Deser A and Waldron A 2013 *Phys.Rev.Lett.* **110** 111101
- [33] Hassan S F and Schmidt-May A and von Strauss M 2013 Higher derivative gravity and conformal gravity from bimetric and partially massless bimetric theory. *Preprint* arXiv:1303.6940
- [34] Hassan S F and Schmidt-May A and von Strauss M 2012 On consistent theories of massive spin-2 fields coupled to gravity. *Preprint* arXiv:1208.1515
- [35] Baccetti V and Martin-Moruno P and Visser M 2012 *JHEP* **1208** 148
- [36] Gumrukcuoglu A E and Lin C and Mukohyama S 2011 *JCAP* **1111** 030

- [37] De Felice A and Gumrukcuoglu E A and Mukohyama S 2012 *Phys.Rev.Lett.* **109** 171101
- [38] Baccetti V and Martin-Moruno P and Visser M 2012 *JHEP* **1208** 108
- [39] Comelli D and Crisostomi M and Pilo L 2012 *JHEP* **1206** 085
- [40] Berg M and Buchberger I and Enander J and Mortsell E and Sjors S 2012 *JCAP* **1212** 021
- [41] Khosravi N and Sepangi H R and Shahidi S 2012 *Phys.Rev.* **D86** 043517
- [42] Akrami Y and Koivisto T S and Sandstad M 2013 *JHEP* **1303** 099
- [43] Belinskii V A and Lifshitz E M and Khalatnikov I M 1972 *Sov.JETP* **35** 838
- [44] Isham C J and Storey D 1978 *Phys.Rev.* **D18** 1047
- [45] Cai Y F and Easson D A and Gao C and Saridakis E N 2013 *Phys.Rev.* **D87** 064001
- [46] Mirbabayi M and Gruzinov A 2013 Black hole discharge in massive electrodynamics and black hole disappearance in massive gravity. *Preprint* arXiv:1302.2665
- [47] Berezhiani L and Chkareuli G and de Rham C and Gabadadze G and Tolley A J 2012 *Phys.Rev.* **D85** 044024
- [48] Deffayet C and Jacobson T 2012 *Class.Quant.Grav.* **29** 065009
- [49] Babichev E and Deffayet C and Ziour R 2009 *Phys.Rev.Lett.* **103** 201102
- [50] Babichev E and Deffayet C and Ziour R 2010 *Phys.Rev.* **D82** 104008
- [51] Gruzinov A and Mirbabayi M 2011 *Phys.Rev.* **D84** 124019
- [52] Gumrukcuoglu E A Lin C and Mukohyama S 2012 *JCAP* **1203** 006
- [53] D'Amico G 2012 *Phys.Rev.* **D86** 124019
- [54] Chiang C and Izumi K and Chen P 2012 *JCAP* **1212** 025
- [55] Wyman M and Hu W and Gratia P 2012 Self-accelerating massive gravity: time for field fluctuations. *Preprint* arXiv:1211.4576
- [56] Cardone V F and Radicella N and Parisi L 2012 *Phys.Rev.* **D85** 124005
- [57] Parisi L and Radicella N and Vilasi G 2012 *Phys.Rev.* **D86** 024035
- [58] Cai Y F and Gao C and Saridakis E N 2012 *JCAP* **1210** 048
- [59] Saridakis E N 2013 *Class.Quant.Grav.* **30** 075003
- [60] Hinterbichler K and Stokes J and Trodden M 2013 Cosmologies of extended massive gravity. *Preprint* arXiv:1301.4993
- [61] Huang Q G and Piao Y S and Zhou S Y 2012 *Phys.Rev.* **D86** 124014
- [62] D'Amico G and Gabadadze G and Hui L and Pirtskhalava D 2012 Quasi-dilaton: theory and cosmology. *Preprint* arXiv:1206.4253
- [63] Haghani Z and Sepangi H R and Shahidi S 2013 Cosmological perturbations of quasi-dilaton non-linear massive gravity. *Preprint* arXiv:1303.2843
- [64] Langlois D and Naruko A 2012 *Class.Quant.Grav.* **29** 202001
- [65] Gong Y 2012 *Commun.Theor.Phys.* **59** 319
- [66] Motohashi H and Suyama T 2012 *Phys.Rev.* **D86** 081502
- [67] Gumrukcuoglu E A and Lin C and Mukohyama S 2012 *Phys.Lett.* **B717** 295
- [68] De Felice A and Gumrukcuoglu A E and Lin C and Mukohyama S 2013 Nonlinear stability of cosmological solutions in massive gravity. *Preprint* arXiv:1303.4154
- [69] Sakaihara Y and Soda J and Takahashi T 2013 On cosmic no-hair in bimetric gravity and the Higuchi bound. *Preprint* arXiv:1211.5976
- [70] Gabadadze G and Giribet G and Iglesias A 2012 New massive gravity on de Sitter space and black holes at the special point. *Preprint* arXiv:1212.6279
- [71] de Rham C and Renaux-Petel S 2013 *JCAP* **1301** 035
- [72] Fasiello M and Tolley A J 2012 *JCAP* **1211** 035
- [73] de Rham C and Hinterbichler K and Rosen R A and Tolley A J 2013 Evidence for and obstructions to non-linear partially massless gravity. *Preprint* arXiv:1302.0025
- [74] Higuchi A 1987 *Nucl.Phys.* **B282** 397
- [75] Zhang Y and Saito R and Sasaki M 2013 *JCAP* **1302** 029
- [76] Park M and Sorbo L 2013 *Phys.Rev.* **D87** 024041